To evaluate a line integral of the form $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{r} = \langle x(t), y(t), z(t) \rangle, a \leq t \leq b$:

- set $\mathbf{r}(t) = \langle x, y, z \rangle$
- write $\mathbf{F}$ in terms of t
- evaluate $\int_a^b \mathbf{F} \cdot \mathbf{r}'(t) dt$

(WW 16) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle 4x, 3y, -2z \rangle$ and C is given by the vector function $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle, 0 \leq t \leq 3\pi/2$.

We know $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$, so equating component functions gives

$$x = \sin t, y = \cos t, z = t.$$

This means that

$$\mathbf{F} = \langle 4x, 3y, -2z \rangle = \langle 4 \sin t, 3 \cos t, -2t \rangle \text{ and}$$

$$\mathbf{r}'(t) = \langle \cos t, -\sin t, 1 \rangle, \text{ so}$$

$$\mathbf{F} \cdot \mathbf{r}'(t) = 4 \sin t \cos t - 3 \sin t \cos t - 2t = \sin t \cos t - 2t.$$

So the relevant integral is

$$\int_0^{3\pi/2} \sin t \cos t - 2t \, dt$$

Recall that $2 \sin t \cos t = \sin(2t)$, so $\sin t \cos t = \frac{1}{2} \sin(2t)$. So the integral is

$$\int_0^{3\pi/2} \frac{1}{2} \sin(2t) - 2t \, dt = \left(-\frac{1}{4} \cos(2t) - t^2 \right) \Big|_{t=0}^{t=3\pi/2} = \frac{2 - 9\pi^2}{4}.$$

Earlier we talked about conservative vector fields. Well why do we care if they're conservative or not? Setting up line integrals can take a lot of computation, but it turns out that if $\mathbf{F}$ is a conservative vector field, then after finding a potential function f (i.e., $\mathbf{F} = \nabla f$), then the value of the line integral from A to B is the same for ANY curve, and it can be computed by $f(B) - f(A)$. *This is the fundamental theorem of Calculus for line integrals.*

(WW 13) Let $\mathbf{F} = \langle x, y \rangle$. Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is the parabola $x = t, y = t^2, 0 \leq t \leq 3$ and C_2 is the line segment $x = 3t^2, y = 9t^2, 0 \leq t \leq 1$.

In both cases, the start and endpoints of the curve are $(0, 0)$ and $(3, 9)$. By inspection, $\mathbf{F}$ is conservative and a potential is

$$f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

So by the previous discussion, the line integral for BOTH paths is

$$f(3, 9) - f(0, 0) = 90/2 - 0/2 = 45.$$

You can check this by parametrizing both curves and in both cases the answer will be 45.